in 1976, and Mary Jones (of Iceland) in 1977 (with respective mothers, Mrs Smith and Mrs Jones). But according to Miller's proposal we could equally regard Miss World as having always been the *same person* who changes through time. It will follow from this that there is a person, namely Miss World, who has a different mother each year. That there are several mothers who have given birth to one and the same child, Miss World, seems to be a rather embarrassing biological consequence of Miller's thesis together with a couple of universally uncontested facts. Miller could retort that *the mother of Miss World* is indeed also a single person, but one who herself changes. One year she is Mrs Smith, and another she happens to be Mrs Jones. But then it would follow that each year, at the instant that Miss World is crowned, there is someone in the world, the mother of Miss World, who instantaneously changes her position—her change of position would be instantaneous and discontinuous. This would contradict the theory of relativity—one would have to add exceptions to the theory in order to take care of the amazing capabilities of Miss World's mother. It is perhaps not logically impossible to go on revising the fundamental views we have on people, space and time, but it hardly seems that it is worth the trouble to justify a dislike for the straight rule.

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CAUCHY'S VARIABLES AND ORDERS OF THE INFINITELY SMALL

In his four textbooks, published between 1821 and 1829, Cauchy often uses variables converging to zero, which he calls 'infinitely small quantities' (Cauchy [1821], p. 27). But what are Cauchy's variables?

Cauchy says: 'One calls a quantity variable which one thinks of as having to take on successively a number of values different from each other' (Cauchy [1821], p. 4). This is all he has to say about the definition of variable in his textbooks ([1821], [1823], [1826], [1829]). Abraham Robinson suggested that what Cauchy meant by a variable is 'a function whose range is numerical while its domain may be any ordered set without last element' (Robinson [1966], p. 270). John Cleave says: 'As a first attempt at interpreting [Cauchy's] "variable" in terms of non-standard analysis, we can imagine that the successive values of a variable are prescribed by a sequence $s = \{s(n) : n \in N\}$ of reals (*i.e.* a function in R with domain N' and range in R)' (Cleave [1971], p. 29). According to this, Cauchy's infinitely small quantities become real null sequences.

However, we must be cautious in interpreting Cauchy's variables as functions (or sequences), since Cauchy defines functions in terms of variables (Cauchy [1821], p. 19). Furthermore, it appears that at times Cauchy let his variables take on infinitesimal as well as real values, as I have argued elsewhere (Fisher [1978]). In fact, Cauchy does not make his concept of variable very clear.

My purpose here is to investigate Cauchy's concept of 'variable decreasing indefinitely' or 'infinitely small quantity' by examining the theory of infinitely small quantities he introduced in his *Cours d'Analyse* of 1821. We will see in the process that Cauchy's variables are not what Robinson or Cleave understand by functions or sequences. Cauchy develops his theory of infinitely small quantities further in his later textbooks (Cauchy [1823], appendix; Cauchy [1826], Lecture 9; Cauchy [1829], Lecture 6), and he presents some interesting classifications and applications. However, we will be concerned here only with the foundations of his theory.

At the same time, we will see clearly how Cauchy had to give up some of the simplicity of actual infinitesimals when he worked with his infinitely small quantities. The situation is somewhat analogous to that of Copernicus whose theory, it is said, was at first more complicated than Ptolemy's for many purposes. Some also say that from the point of view of relativity theory, Ptolemy's theory is justifiable—it is only a matter of where one puts the center of coordinates. Similarly, we now know from the work of Abraham Robinson and his followers on non-standard analysis that actual infinitesimals are justifiable, so Cauchy need not have given them up in the name of rigor.

To begin with, we note that if the existence of non-zero infinitesimals is admitted, it is easy enough to compare their sizes. We can simply take their ratios. For example, given a function f we can set

$$dy = f(x+dx)-f(x)$$

for a non-zero infinitesimal dx, and take the derivative of f at x to be dy/dx. A modern definition of this kind is given by Martin Davis (Davis [1977], pp. 63–5). Naturally the case where dy/dx is finite is of special interest. Clearly this will happen if and only if $dy = k\,dx$ for some finite k.

However, Cauchy does not proceed directly with infinitesimals. He introduces instead a calculus of 'infinitely small quantities'. In the *Cours d'Analyse*, he says: 'One says that a variable quantity becomes *infinitely small* when its numerical value decreases indefinitely in such a way as to converge toward the limit zero' (Cauchy [1821], p. 26). And again: 'Let α be an infinitely small quantity, that is, a variable whose numerical [*i.e.* absolute] value decreases indefinitely' (Cauchy [1821], p. 27). Then, Cauchy continues, α, α^2, α^3, *etc.*, are called infinitely small of the first, second, third order, *etc.* It is notable that Cauchy does not explicitly require the values of these variables to be finite.

More generally, says Cauchy, a variable quantity is infinitely small of the first order if its ratio with α converges to a finite limit different from zero as the numerical value of α decreases; infinitely small of the second order if it varies with α and its ratio with α^2 converges to such a limit, *etc.* Presumably Cauchy does not allow α to have the value zero.

'Given this', he continues, 'if one denotes by k a finite quantity different from zero, and by α a variable number which decreases indefinitely with the numerical value of α, the general form of infinitely small quantities of the first order will be $k\alpha$ or at least $k\alpha(1 \pm \epsilon)$' (Cauchy [1821], p. 28). In Cauchy's terminology, a 'number' is positive, so ϵ is always positive. 'The general form', Cauchy continues, 'of the infinitely small [quantities] of order n . . . will be $k\alpha^n$ or at least $k\alpha^n(1 \pm \epsilon)$' (Cauchy [1821], p. 28). (Cauchy speaks here of *infiniment petits* rather than *quantités infiniment petites.*)

Cauchy does not offer any proof that the general forms follow from his definition. Actually, there are a couple of small difficulties. These can be avoided by claiming the general form to be $k\alpha^n(1+\epsilon)$, where ϵ may be negative as well as positive. Then we can supply a proof that this is indeed the most general form. In fact, if $\beta = \beta(\alpha)$ is an infinitely small quantity and β/α^n goes to $k \neq 0$ as α goes to 0 (or $|\alpha|$ goes to 0), set $\epsilon = (\beta/k\alpha^n) - 1$. Then β goes to 0 with α, and $\beta = k\alpha^n(1+\epsilon)$. Conversely, if $\beta = k\alpha^n(1+\epsilon)$ where ϵ goes to 0 with α, then β/α^n goes to k as α goes to 0 (even if $k = 0$).

In any case, Cauchy considers infinitely small quantities of the form $k\alpha^n(1 \pm \epsilon)$. He proves, for example: 'If one compares two infinitely small [quantities] of different orders when both converge to the limit zero, that which has the higher order will finish by always taking on the smaller numerical value' (Cauchy [1821], p. 29).

Cauchy's proof runs as follows:

> In fact, let $k\alpha^n(1 \pm \epsilon)$, $k'\alpha^{n'}(1 \pm \epsilon')$ be two infinitely small [quantities], one of order n and the other of order n', and suppose $n' > n$; the ratio between the second of these infinitely small [quantities] and the first, namely,
>
> $$(k'/k)\alpha^{n'-n}((1 \pm \epsilon')/(1 \pm \epsilon))$$
>
> will converge indefinitely with α toward the limit zero; which cannot happen unless the numerical value of the second ends by becoming always less than that of the first (Cauchy [1821], p. 29).

This is all the proof Cauchy gives, and it is not very explicit. He can perhaps be taken to have meant when he says 'the numerical value of the second ends by becoming always less than the first' that there is a number t such that

$$|k'\alpha^{n'}(1 \pm \epsilon')| < |k\alpha^n(1 \pm \epsilon)|$$

for all $|\alpha| < t$. Whether we should take t finite or not is a moot question; Cauchy gives no hint. In any case, given this as equivalent to Cauchy's conclusion, we can fill in a proof as follows. If this conclusion were not so, there would be a sequence $\alpha_1, \alpha_2, \ldots$ going to zero such that

$$|k'\alpha_j^{n'}(1 \pm \epsilon')| \geqslant |k\alpha_j^n(1 \pm \epsilon)|$$

for each j. Then we would have

$$|(k'\alpha_j^{n'}(1 \pm \epsilon'))/(k_j^n(1 \pm \epsilon)|. \geqslant 1$$

for all j. Yet $|\alpha_j^{n'-n}(\mathrm{k}'/k)((1 \pm \epsilon')/(1 \pm \epsilon))|$ must go to zero as $\alpha_1, \alpha_2, \ldots$ does, since $n' - n > 0$.

At this stage, even though Cauchy does not say we should, it is easy to interpret his infinitely small quantities as variables which go in some way not further specified to zero, while taking on only finite (real) values, *i.e.* without taking on infinitesimal values. But we see the price Cauchy had to pay. The theorem about infinitely small quantities we just quoted corresponds to the following theorem about infinitesimals: If α is an infinitesimal and $n' > n$, then $\alpha n'/\alpha^n$ is an infinitesimal.

Let us now examine a little how Cauchy applies his infinitely small quantities. A theorem on the next page from the one we quoted above says:

> Any polynomial ordered according to increasing powers of α, for example, $a + b\alpha + c\alpha^2 + \&c \ldots$ or more generally,
>
> $$a\alpha^n + b\alpha^{n'} + c\alpha^{n''} + \&c \ldots$$
>
> (the numbers n, n', n'', ... forming an increasing sequence) ends by being always of the same sign as its first term a or $a\alpha^n$ for very small values of α (Cauchy [1821], pp. 30–1).

Cauchy's proof is as follows.

> In fact, the sum of the second term and those which follow it is, in the first case, an infinitely small quantity of the first order whose numerical value ends by being less than that of the finite quantity a and, in the second case, an infinitely small quantity of order n', which ends by always having a numerical value less than that of an infinitely small quantity of order n (Cauchy [1821], pp. 31).

It is rather curious that Cauchy says of $b\alpha + c\alpha^2 + \&c \ldots$ that its 'numerical value ends by being less than that of the finite quantity *a*', using the singular 'value'. But perhaps this can be taken as a locution for the plural 'numerical values end by being less than ... *a*'. The second part of this theorem follows from the first theorem we quoted. But as we have seen, Cauchy did not really so much prove the earlier theorem as declare it. He cannot be said to have used reasoning about sequences, or about limits in an epsilon-delta manner, although he sometimes did use such reasoning in other contexts (see Fisher [1978]).

A short space later, we find the following theorem:

> If, in the polynomial $a + b\alpha^{n'} + c\alpha^{n''} + \&c \ldots$ ordered according to increasing powers of α, n' denotes an even number, then among the values of this polynomial corresponding to infinitely small values of α, that which corresponds to $\alpha = 0$, that is, a, will always be the smallest when b is positive, and the greatest when b is negative (Cauchy [1821], p. 32).

Cauchy adds: 'This particular value of the polynomial, greater or smaller than all the neighboring values, is what one calls a *maximum* or *minimum*'.

Now we are some distance from the language of finite real variables. It is one thing to speak of the values of a polynomial $p(\alpha)$ for real values of α such that $|\alpha|$ is less than some positive real number t. But it is another thing to speak of values of $p(\alpha)$ corresponding to infinitely small values of α. This is close to the language of actual infinitesimals.

We have seen how Cauchy used his infinitely small quantities immediately after he introduced them. Their nature becomes no clearer later on in his textbooks. In the forwards to both his [1823] and his [1829], he says:

> My principal aim has been to reconcile the rigor, which I made my law in my *Cours d'Analyse*, with the simplicity which results from the direct consideration of infinitely small quantities (Cauchy [1823], p. v; [1829], p. 1 of the reprinting in his *Oeuvres*).

Cauchy does not say what he means by 'direct consideration of infinitely small quantities'. Perhaps he expected his readers to know what he meant.

We must conclude, I think, that Cauchy did not make very clear what he meant by his infinitely small quantities, or variables with limit zero. It is quite clear, however, that he did not understand them to be functions or sequences, in the senses in which we now take these words.

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